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2010 J. Phys. A: Math. Theor. 43 055204

(http://iopscience.iop.org/1751-8121/43/5/055204)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **43** (2010) 055204 (12pp) doi:10.1088/1751-8113/43/5/055204

The general time fractional wave equation for a vibrating string

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Received 30 June 2009, in final form 7 December 2009
Published 14 January 2010
Online at stacks.iop.org/JPhysA/43/055204

Abstract
The solution of a general time fractional wave equation for a vibrating string is obtained in terms of the Mittag–Leffler-type functions and complete set of eigenfunctions of the Sturm–Liouville problem. The time fractional derivative used is taken in the Caputo sense, and the method of separation of variables and the Laplace transform method are used to solve the equation. Some results for special cases of the initial and boundary conditions are obtained and it is shown that the corresponding solutions of the integer order equations are special cases of those of time fractional equations. The proposed general equation may be used for modeling different processes in complex or viscoelastic media, disordered materials, etc.

PACS numbers: 45.10.Hj, 02.30.Gp

1. Introduction

In this paper we investigate the general time fractional partial differential equation of the form

$$r(x)D_*^{\alpha}u(x,t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u(x,t)}{\partial x}\right] - q(x)u(x,t) + f(x,t), \qquad t>0, \quad 0\leqslant x\leqslant l \tag{1}$$

$$\left[b_1\frac{\partial u(x,t)}{\partial x} + a_1u(x,t)\right]\Big|_{x=0} = h_1(t), \qquad \left[b_2\frac{\partial u(x,t)}{\partial x} + a_2u(x,t)\right]\Big|_{x=l} = h_2(t) \tag{2}$$

$$\frac{\partial^k u(x,t)}{\partial t^k}\Big|_{t=0+} = g_k(x), \qquad k=0,1,\ldots,m-1, \qquad m-1<\alpha\leqslant m, \tag{3}$$

where D_*^α is the time fractional differential operator in the Caputo sense ($\alpha > 0$), $p(x) > 0$, $r(x) > 0$ and $q(x)$ are given continuous functions in $[0, l]$, $f(x, t)$, $h_1(t)$, $h_2(t)$ and $g_k(x)$ are given sufficiently well-behaved functions, and a_1, a_2, b_1 and b_2 are constants. The time fractional differential operator of order $\gamma > 0$ in the Caputo sense is the operator $D_*^\gamma f(t)$ such that [3]

$$D_*^\gamma f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\gamma+1-m}}\,\mathrm{d}\tau & \text{if} \quad m-1 < \gamma < m, \\ \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m} & \text{if} \quad \gamma = m. \end{cases} \tag{4}$$

The time fractional differential equation (1), as well as the boundary conditions (2) and initial conditions (3), is very general, and many problems that are already studied are special cases of it. So their solutions can be obtained by changing given functions or constants in the solution of equation (1).

The solution of this problem will be obtained in a bounded domain $x \in [0, l]$ and in the space of summable Lebesgue integrable functions

$$L(0, \infty) = \left\{ f : \|f\|_1 = \int_0^\infty |f(t)|\,\mathrm{d}t < \infty \right\}. \tag{5}$$

In a number of papers many authors have investigated time fractional relaxation and oscillation processes as well as time fractional diffusive and wave processes [12, 20–22]. The fractional Brownian motion, fractional Langevin and fractional Fokker–Planck equations have attracted attention in the last few years [9–11, 13, 18, 23]. The fractional diffusion-wave equation was introduced in physics to describe the diffusion process in media with fractional geometry [27], the fractional diffusive waves in viscoelastic solids which exhibit a power-law creep [19], the anomalous diffusion and relaxation which are found in diverse physical systems such as charge transport in disordered semiconductors and quantum dots, protein relaxation dynamics, electrochemistry, biomedicine, etc [5, 8, 24, 34, 37]. In many papers the properties of the fractional integral and differential operators as well as the Mittag–Leffler functions are investigated [14, 31] and the results are used for modeling different physical phenomena.

This paper is organized as follows. In section 2 we present the mathematical background related to the fractional differentiation and integration, definitions of the Mittag–Leffler functions and some basic properties. In section 3 we present the technique of solving equation (1) with the boundary conditions (2) and initial conditions (3) by using the method of separation of variables and the Laplace transform method, and several special cases of equation (1) are considered. The conclusion is provided in section 4.

2. Mathematical background

2.1. The Mittag—Leffler function

The Mittag–Leffler function which is introduced by Mittag–Leffler [25] is an entire function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{6}$$

where ($z \in \mathbb{C}$; $\Re[\alpha] > 0$). Wiman [36], Agarwal [1], Humbert [15], Humbert and Agarwal [16], etc, investigated a more general Mittag–Leffler function defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{7}$$

where $z, \beta \in \mathbb{C}; \Re[\alpha] > 0$. The Mittag–Leffler function (7) is an entire function of order $\rho = 1/\Re[\alpha]$ and type 1. Note that $E_{\alpha,1}(z) = E_\alpha(z)$. The Mittag–Leffler function is a generalization of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = \mathrm{e}^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$. For the Mittag–Leffler functions the following formula is true [7]:

$$\int_0^x t^{\alpha-1} E_{\alpha,\alpha}(-at^\alpha)(x-t)^{\beta-1} E_{\alpha,\beta}(-b(x-t)^\alpha)\,\mathrm{d}t = \frac{E_{\alpha,\beta}(-bx^\alpha) - E_{\alpha,\beta}(-ax^\alpha)}{a-b} x^{\beta-1} \qquad (a \neq b). \tag{8}$$

The following Laplace transform formula, which includes the Mittag–Leffler function (7), is very important for solving fractional differential equations [29, 30]:

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)] = \int_0^\infty \mathrm{e}^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)\,\mathrm{d}t = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}, \tag{9}$$

where $\Re[s] > |a|^{1/\alpha}$. Some computations of the Mittag–Leffler function are given by Hilfer and Seybold in the complex plane [14].

2.2. *Fractional integral operator*

The fractional integral of order $\gamma > 0$ is defined as

$$J^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau)\,\mathrm{d}\tau, \qquad t > 0, \tag{10}$$

where J^γ is the so-called fractional integral operator. To complete the definition (10), it is used that $J^0 f(t) = f(t)$. The integral operator (10) is closely connected with the Caputo time fractional differential operator (4). From the definition of the fractional integral (10) it follows that [28]

$$J^\gamma J^\delta = J^{\gamma+\delta} = J^\delta J^\gamma \quad \text{(semi-group property)} \tag{11}$$

$$J^\gamma t^s = \frac{\Gamma(s+1)}{\Gamma(s+1+\gamma)} t^{s+\gamma}, \qquad \gamma \geqslant 0, \quad s > -1, \quad t > 0 \tag{12}$$

and ([17], p 78)

$$(J^\gamma t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu))(x) = x^{\gamma+\beta-1} E_{\mu,\gamma+\beta}(\lambda x^\mu)$$
$$(\lambda \in C, \Re[\gamma] > 0, \Re[\beta] > 0, \Re[\mu] \geqslant 0). \tag{13}$$

Recently Srivastava and Tomovski introduced an integral operator $\left(\mathcal{E}^{\omega;\gamma,\kappa}_{a+;\alpha,\beta}\varphi\right)(x)$ defined as [33]

$$\left(\mathcal{E}^{\omega;\gamma,\kappa}_{a+;\alpha,\beta}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\kappa}_{\alpha,\beta}(\omega(x-t)^\alpha)\varphi(t)\,\mathrm{d}t. \tag{14}$$

In relation (14), $E^{\gamma,\kappa}_{\alpha,\beta}(z)$ is the generalized Mittag–Leffler function which has the following form:

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \tag{15}$$

where $z, \beta, \gamma \in \mathbb{C}; \Re[\alpha] > \max\{0, \Re[\kappa] - 1\}; \Re[\kappa] > 0$ and $(\gamma)_{\kappa n}$ is a notation of the Pochhammer symbol. In the case when $\omega = 0$, the integral operator (14) would correspond to the integral operator (10). Srivastava and Tomovski gave some compositional properties for $E^{\gamma,\kappa}_{\alpha,\beta}(z)$ and solved some linear and nonlinear fractional differential equations. The Mittag–Leffler function $E^\gamma_{\alpha,\beta}(z)$ is a special case of the function (15) for which $E^{\gamma,1}_{\alpha,\beta}(z) = E^\gamma_{\alpha,\beta}(z)$ ([17], p 45) is satisfied. Note that $E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

2.3. Fractional differential operator

From relation (4), it can be easily shown that $D_*^\gamma 1 \equiv 0, \gamma > 0$. The Laplace transform for the Caputo time fractional differential operator is given by the following formula [30]:

$$\mathcal{L}[D_*^\gamma f(t)] = \int_0^\infty \mathrm{e}^{-st} D_*^\gamma f(t)\,\mathrm{d}t = s^\gamma F(s) - \sum_{k=0}^{m-1} f^{(k)}(0+) s^{\gamma-1-k} \qquad (m-1 < \gamma \leqslant m), \tag{16}$$

where $F(s)$ is the Laplace transform of the function $f(t)$.

3. Solution of the general time fractional wave equation for a vibrating string

The partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{x^2} \tag{17}$$

represents a simple wave equation for an elastic string with the constant mass density ρ. Here $a = \sqrt{\frac{T}{\rho}}$, and T is a constant string tension. This wave equation is followed by boundary conditions that can be expressed in different forms, and initial conditions $u(x,0)$ and $\frac{\partial u(x,0)}{\partial t}$, which represent the initial shape of the string and the initial velocity of the string, respectively. For example, the boundary conditions can be expressed as $u(0,t) = 0$ and $u(l,t) = 0$ which means that the ends of the elastic string are fixed. Additional terms can be added to this equation, such as external force which depends on x and t, or mass density which depends on the coordinate x, etc. Many authors have investigated this problem taking the time fractional differential operator instead of the integer order differential operator.

Equation (1) with the boundary conditions (2) and initial conditions (3) is very general and many problems that are already studied are special cases of it. For example, the considered problems in [2, 26] can be obtained if we substitute $0 < \alpha \leqslant 1, r(x) = 1, p(x) = 1, q(x) = 0,$ $f(x,t) = 0$, $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $h_1(t) = h_2(t) = 0$ and $g_0(x) = f(x)$ in relations (1), (2) and (3). The equation in [38] is similar to those in [2] and [26] with a difference that $p(x) = \text{const}$ and $f(x) = x(1-x)$ is used. The equation in [4] is a special case of equation (1) and can be obtained if $0 < \alpha \leqslant 1$, $r(x) = 1$, $p(x) = 1$, $q(x) = 0$, $f(x,t) = 0$, $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $h_1(t) = \varphi(t)$, $h_2(t) = \psi(t)$ and $g_0(x) = w(x)$, and $1 < \alpha \leqslant 2$, $r(x) = 1$, $p(x) = 1$, $q(x) = 0$, $f(x,t) = 0$, $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $h_1(t) = h_2(t) = 0$, $g_0(x) = f(x)$ and $g_1(x) = 0$ in [6] are used. The results from this paper are in fact contained in the following theorem.

Theorem 1. *The general time fractional wave equation for a vibrating string of form (1) with boundary conditions (2) and initial conditions (3) in the case when $1 < \alpha \leqslant 2$ has a summable solution in the space $L(0,\infty)$ with respect to t:*

$$u(x,t) = \sum_{n=1}^\infty a_n(t) X_n(x) + \sum_{n=1}^\infty \left(\mathcal{E}_{0+;\alpha,\alpha}^{-\lambda_n;1,1} \widetilde{f}_n\right)(t) X_n(x) + v(x,t), \tag{18}$$

where $x \in [0,l]$,

$$v(x,t) = \frac{a_2 x - b_2 - a_2 l}{a_2 b_1 - a_1 b_2 - a_1 a_2 l} h_1(t) + \frac{b_1 - a_1 x}{a_2 b_1 - a_1 b_2 - a_1 a_2 l} h_2(t), \tag{19}$$

$$a_n(t) = T_n^{(0)}(0+) E_\alpha(-\lambda_n t^\alpha) + T_n^{(1)}(0+) t E_{\alpha,2}(-\lambda_n t^\alpha), \tag{20}$$

λ_n and $X_n(x) \in L^2[0, l]$ for $n = 1, 2, \ldots$ are eigenvalues and eigenfunctions of the problem, respectively,

$$T_n^{(k)}(0+) = \frac{1}{\|X_n(x)\|^2} \int_0^l \left[g_k(x) - \frac{\partial^k v(x, 0+)}{\partial t^k}\right] X_n(x)\,\mathrm{d}x, \tag{21}$$

for $k = 0, 1$,

$$\widetilde{f}_n(t) = \frac{1}{\|X_n(x)\|^2} \int_0^l \widetilde{f}(x, t) X_n(x)\,\mathrm{d}x \tag{22}$$

and

$$\widetilde{f}(x, t) = f(x, t) + \frac{\partial}{\partial x}\left[p(x)\frac{\partial v(x, t)}{\partial x}\right] - q(x)v(x, t) - r(x)D_*^\alpha v(x, t). \tag{23}$$

Proof. To solve equation (1) with the boundary conditions (2) and initial conditions (3), we represent the function $u(x, t)$ in the form

$$u(x, t) = U(x, t) + v(x, t). \tag{24}$$

The function $v(x, t)$ is chosen to satisfy the boundary conditions (2) of equation (1):

$$\left.\left[b_1\frac{\partial v(x, t)}{\partial x} + a_1 v(x, t)\right]\right|_{x=0} = h_1(t), \qquad \left.\left[b_2\frac{\partial v(x, t)}{\partial x} + a_2 v(x, t)\right]\right|_{x=l} = h_2(t). \tag{25}$$

It can be easily obtained that the function $v(x, t)$ has the from (19). From relations (25) and (24) for the function $U(x, t)$ one obtains

$$\left.\left[b_1\frac{\partial U(x, t)}{\partial x} + a_1 U(x, t)\right]\right|_{x=0} = 0, \qquad \left.\left[b_2\frac{\partial U(x, t)}{\partial x} + a_2 U(x, t)\right]\right|_{x=l} = 0. \tag{26}$$

From the initial conditions (3) and by using relation (24) it can be obtained that

$$\left.\frac{\partial^k U(x, t)}{\partial t^k}\right|_{t=0+} = g_k(x) - \left.\frac{\partial^k v(x, t)}{\partial t^k}\right|_{t=0+} = \widetilde{g}_k(x) \tag{27}$$

for $k = 0, 1, \ldots, m-1$ and $m - 1 < \alpha \leqslant m$.

By using the substitution

$$U(x, t) = U_1(x, t) + U_2(x, t) \tag{28}$$

from relations (1), (24) and (28), it follows that

$$r(x)D_*^\alpha[U_1(x, t) + U_2(x, t)] = \left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] - q(x)\right\}[U_1(x, t) + U_2(x, t)] + \widetilde{f}(x, t), \tag{29}$$

where $\widetilde{f}(x, t)$ is given by (23).

The functions in relation (29) can be separated in the following way:

$$r(x)D_*^\alpha U_1(x, t) = \left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] - q(x)\right\} U_1(x, t), \tag{30}$$

$$\left.\left[b_1\frac{\partial U_1(x, t)}{\partial x} + a_1 U_1(x, t)\right]\right|_{x=0} = 0, \qquad \left.\left[b_2\frac{\partial U_1(x, t)}{\partial x} + a_2 U_1(x, t)\right]\right|_{x=l} = 0 \tag{31}$$

$$\left.\frac{\partial^k U_1(x, t)}{\partial t^k}\right|_{t=0+} = \widetilde{g}_k(x) \tag{32}$$

for $k = 0, 1, \ldots, m-1$ and $m - 1 < \alpha \leqslant m$ and

$$r(x)D_*^\alpha U_2(x,t) = \left\{\frac{\partial}{\partial x}\left[p(x)\frac{\partial}{\partial x}\right] - q(x)\right\} U_2(x,t) + \widetilde{f}(x,t), \tag{33}$$

$$\left[b_1\frac{\partial U_2(x,t)}{\partial x} + a_1U_2(x,t)\right]\bigg|_{x=0} = 0, \qquad \left[b_2\frac{\partial U_2(x,t)}{\partial x} + a_2U_2(x,t)\right]\bigg|_{x=l} = 0, \tag{34}$$

$$\frac{\partial^k U_2(x,t)}{\partial t^k}\bigg|_{t=0+} = 0 \tag{35}$$

for $k = 0, 1, \ldots, m-1$ and $m-1 < \alpha \leqslant m$.

By using the method of separation of variables in equation (30) representing the function $U_1(x,t)$ as a product of two functions $U_1(x,t) = X(x)T(t)$, one obtains the following differential equations:

$$D_*^\alpha T(t) + \lambda T(t) = 0, \tag{36}$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] - q(x)\right\} X(x) + \lambda r(x)X(x) = 0, \tag{37}$$

where λ is a separation constant. The function $X(x)$ satisfies the following boundary conditions:

$$\left[b_1\frac{\mathrm{d}X(x)}{\mathrm{d}x} + a_1X(x)\right]\bigg|_{x=0} = 0, \qquad \left[b_2\frac{\mathrm{d}X(x)}{\mathrm{d}x} + a_2X(x)\right]\bigg|_{x=l} = 0. \tag{38}$$

Equation (37), with the boundary conditions (38), represents the Sturm–Liouville problem which has a spectrum of eigenvalues λ_n ($\lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots$) and a complete set of eigenfunctions $X_n(x)$ for which, in the Hilbert space $L^2[0,l]$, the following is satisfied:

$$\int_0^l r(x)X_n^2(x)\,\mathrm{d}x = \|X_n(x)\|^2\delta_{nm}. \tag{39}$$

In relation (39) $r(x)$ is the weight or density function, $\|X_n\|^2$ is the norm of the eigenfunction $X_n(x)$ and δ_{nm} is the Kronecker delta. The eigenfunction $X_n(x)$ is called the nth fundamental solution satisfying the regular Sturm–Liouville problem (37) and (38). It is known that in the case when $q(x) \geqslant 0$, all the eigenfunctions are positive [35].

Equation (36) can be solved by using relation (16) for the Laplace transform of the Caputo time fractional differential operator (4). Thus, one obtains

$$s^\alpha\mathcal{L}[T_n(t)] - \sum_{k=0}^{m-1} T_n^{(k)}(0+)s^{\alpha-1-k} + \lambda_n\mathcal{L}[T_n(t)] = 0, \tag{40}$$

from where it is obtained that

$$\mathcal{L}[T_n(t)] = \sum_{k=0}^{m-1} T_n^{(k)}(0+)\frac{s^{\alpha-1-k}}{s^\alpha + \lambda_n}. \tag{41}$$

From relation (41) and by using relation (9) the solution in terms of the Mittag–Leffler function (7)

$$T_n(t) = \sum_{k=0}^{m-1} T_n^{(k)}(0+)t^k E_{\alpha,k+1}(-\lambda_n t^\alpha) \tag{42}$$

is obtained, where $T_n^{(k)}(0+)$ for $k = 0, 1$ are given by (21). So the solution of equation (30) is given by

$$U_1(x,t) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{m-1} T_n^{(k)}(0+)t^k E_{\alpha,k+1}(-\lambda_n t^\alpha)\right) X_n(x). \tag{43}$$

This sum represents the solution of the problem of free oscillations of a vibrating string with non-zero initial conditions. It is a generalized Fourier expansion of the function $U_1(x,t)$ by using the set of eigenfunctions $X_n(x)$ as a basis. Since the function $U_1(x,t)$ satisfies same boundary conditions as those of the eigenfunctions $X_n(x)$, and if we suppose that $\frac{\partial U_1(x,t)}{\partial x}$ is continuous, then expansion (43) converges absolutely and uniformly in the interval $[0,l]$ to the function $U_1(x,t)$ [35].

The solution of equation (33) can be found by using the complete set of eigenfunctions $X_n(x)$:

$$U_2(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x). \tag{44}$$

Also, this sum converges absolutely and uniformly in the interval $[0,l]$ to the function $U_2(x,t)$ since we suppose that $\frac{\partial U_2(x,t)}{\partial x}$ is continuous, and $U_2(x,t)$ satisfies the same boundary conditions as those of the eigenfunctions $X_n(x)$.

Let us expand the function $\widetilde{f}(x,t)$ in the following form:

$$\widetilde{f}(x,t)=\sum_{n=1}^{\infty}\widetilde{f}_n(t)r(x)X_n(x), \tag{45}$$

where $\widetilde{f}_n(t)$ is given by (22). By using relations (44)–(45), (22) and (33), one obtains

$$\sum_{n=1}^{\infty}[D_*^{\alpha}u_n(t)+\lambda_n u_n(t)-\widetilde{f}_n(t)]r(x)X_n(x)=0, \tag{46}$$

which is satisfied if

$$D_*^{\alpha}u_n(t)+\lambda_n u_n(t)-\widetilde{f}_n(t)=0. \tag{47}$$

By applying the Laplace transform method to equation (47) one obtains

$$s^{\alpha}\mathcal{L}[u_n(t)]-\sum_{k=0}^{m-1}u_n^{(k)}(0+)s^{\alpha-1-k}+\lambda_n\mathcal{L}[u_n(t)]-\mathcal{L}[\widetilde{f}_n(t)]=0. \tag{48}$$

From the conditions (35) it follows that $\frac{\partial^k u_n(x,t)}{\partial t^k}|_{t=0+}=0$ for $k=0,1,\ldots,m-1$ and $m-1<\alpha\leqslant m$. Thus, from (48) it follows that

$$\mathcal{L}[u_n(t)]=\frac{1}{s^{\alpha}+\lambda_n}\mathcal{L}[\widetilde{f}_n(t)]=\mathcal{L}[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})]\mathcal{L}[\widetilde{f}_n(t)]. \tag{49}$$

From relation (49) it can be noticed that $u_n(t)$ is a convolution of two functions, i.e.

$$u_n(t)=\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\widetilde{f}_n(\tau)\,\mathrm{d}\tau. \tag{50}$$

So, the solution of equation (33) in terms of the Mittag–Leffler function is

$$U_2(x,t)=\sum_{n=1}^{\infty}\left[\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\widetilde{f}_n(\tau)\,\mathrm{d}\tau\right]X_n(x). \tag{51}$$

Solution (51) can be expressed by using the integral operator (14). So it becomes

$$U_2(x,t)=\sum_{n=1}^{\infty}\left(\mathcal{E}_{0+;\alpha,\alpha}^{-\lambda_n;1,1}\widetilde{f}_n\right)(t)X_n(x). \tag{52}$$

This sum represents the oscillations of the vibrating string in the presence of an external force and zero initial conditions.

Finally, in the case when $1 < \alpha \leqslant 2$, by using relations (24), (28), (43) and (52), we get (18) which finishes the proof of theorem 1. Note that if we put $\alpha = 2$ in equation (1) we obtain the general integer order wave equation for a vibrating string [35]. From solution (18), by appropriate substitution of the functions and coefficients from relations (1), (2) and (3), the results obtained in [2, 4, 6, 26, 38] are easily followed. □

Corollary 1. *The time fractional partial differential equation*

$$D_*^\alpha u(x,t) = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + b \sin x, \tag{53}$$

with boundary conditions

$$u(x,t)|_{x=0} = 0, \qquad u(x,t)|_{x=l} = 0 \tag{54}$$

and initial conditions

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+} = 0, \qquad u(x,0+) = g(x), \tag{55}$$

where $1 < \alpha < 2$, $0 \leqslant x \leqslant l$, b is a constant, has a solution of the form

$$\begin{aligned} u(x,t) = {} & \sum_{n=1}^{\infty} c_n E_\alpha\left(-\frac{n^2\pi^2a^2}{l^2}t^\alpha\right)\sin\left(\frac{n\pi x}{l}\right) \\ & + 2b\pi \sin l \sum_{n=1}^{\infty} \frac{(-1)^n n}{l^2 - n^2\pi^2} t^\alpha E_{\alpha,\alpha+1}\left(-\frac{n^2\pi^2a^2}{l^2}t^\alpha\right)\sin\left(\frac{n\pi x}{l}\right), \end{aligned} \tag{56}$$

where c_n is the Fourier coefficient given by

$$c_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x. \tag{57}$$

Proof. This equation is a special case of equation (1) where $f(x,t) = b\sin x$, $r(x) = 1$, $p(x) = a^2$, $q(x) = 0$, $h_1(t) = h_2(t) = 0$, $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. By substituting these values in solution (43) we obtain the first term of the relation (56). From relations (18)–(23) one obtains

$$f_n(t) = \frac{2}{l}\int_0^l \sin x \sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x = 2\pi \sin l \frac{(-1)^n n}{l^2 - n^2\pi^2}, \tag{58}$$

from where for the solution $U_2(x,t)$ it follows that

$$U_2(x,t) = \sum_{n=1}^{\infty} 2b\pi \sin l \frac{(-1)^n n}{l^2 - n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{n^2\pi^2a^2}{l^2}\tau^\alpha\right)\mathrm{d}\tau. \tag{59}$$

By substituting relation (13) ($\gamma = 1$) in relation (59) we obtain solution (56). □

Corollary 2. *The time fractional partial differential equation*

$$D_*^\alpha u(x,t) = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{60}$$

with boundary conditions

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = 0 \tag{61}$$

and initial conditions

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+} = 0, \qquad u(x,0+) = g(x), \tag{62}$$

where $1 < \alpha < 2$ *and* $0 \leqslant x \leqslant l$ *has a solution of the form*

$$u(x,t) = \sum_{n=1}^{\infty} c_n E_\alpha\left(-\frac{n^2\pi^2 a^2}{l^2}t^\alpha\right)\sin\left(\frac{n\pi x}{l}\right), \tag{63}$$

where c_n *is given by relation (57).*

Proof. This equation is a special case of equation (53) where $b = 0$, so solution (63) is directly obtained from solution (56).

Note that for $\alpha = 2$, solution (63) has the following well-known form:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi a t}{l}\right)\sin\left(\frac{n\pi x}{l}\right). \tag{64}$$

□

Corollary 3. *The ftime fractional partial differential equation*

$$D_*^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + ct^{\gamma-1}E_{\alpha,\gamma}(-bt^\alpha), \tag{65}$$

with boundary conditions

$$u(x,t)|_{x=0} = 0, \qquad u(x,t)|_{x=l} = 0 \tag{66}$$

and initial conditions

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+} = 0, \qquad u(x,0+) = g(x), \tag{67}$$

where $1 < \alpha < 2$, $0 \leqslant x \leqslant l$, $1 < \gamma < 2$, *b and c are constants, has a solution of the form*

$$u(x,t) = \sum_{n=1}^{\infty} c_n E_\alpha\left(-\frac{n^2\pi^2}{l^2}t^\alpha\right)\sin\left(\frac{n\pi x}{l}\right) + \\ + 2ct^{\gamma-1}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\cdot\frac{E_{\alpha,\gamma}(-bt^\alpha) - E_{\alpha,\gamma}\left(-\frac{n^2\pi^2}{l^2}t^\alpha\right)}{\frac{n^2\pi^2}{l^2} - b}\sin\left(\frac{n\pi x}{l}\right). \tag{68}$$

Proof. This equation is a special case of equation (1) where $f(x,t) = ct^{\gamma-1}E_{\alpha,\gamma}(-bt^\alpha)$, $r(x) = 1$, $p(x) = 1$, $q(x) = 0$, $h_1(t) = h_2(t) = 0$, $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. The first term of relation (68) is the same as the first term of the solution in corollary 1. From relations (18)–(23) one obtains

$$f_n(t) = \frac{2}{l}\int_0^l ct^{\gamma-1}E_{\alpha,\gamma}(-bt^\alpha)\sin\left(\frac{n\pi x}{l}\right)\mathrm{d}x = \frac{2\,[1-(-1)^n]}{n\pi}ct^{\gamma-1}E_{\alpha,\gamma}(-bt^\alpha). \tag{69}$$

Thus, for the solution $U_2(x,t)$ it follows that

$$U_2(x,t) = \sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}cI_{\alpha,\gamma,b,n}(t)\sin\left(\frac{n\pi x}{l}\right), \tag{70}$$

where the integral $I_{\alpha,\gamma,b,n}(t)$ is given by

$$I_{\alpha,\gamma,b,n}(t) = \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)(t-\tau)^{\gamma-1}E_{\alpha,\gamma}(-b(t-\tau)^\alpha)\,\mathrm{d}\tau. \tag{71}$$

Using relation (8) with $a = \lambda_n$, $\beta = \gamma$ we get

$$I_{\alpha,\gamma,b,n}(t) = t^{\gamma-1}\frac{E_{\alpha,\gamma}(-bt^\alpha) - E_{\alpha,\gamma}(-\lambda_n t^\alpha)}{\lambda_n - b}. \tag{72}$$

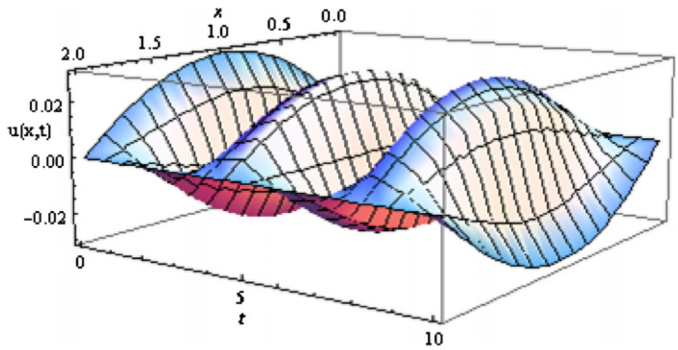

Figure 1. Graphical representation of solution (78) for $a = 1$.
(This figure is in colour only in the electronic version)

Thus, equation (70) becomes

$$U_2(x,t) = 2ct^{\gamma-1}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\cdot\frac{E_{\alpha,\gamma}(-bt^{\alpha}) - E_{\alpha,\gamma}(-\lambda_n t^{\alpha})}{\lambda_n - b}\sin\left(\frac{n\pi x}{l}\right), \tag{73}$$

from where solution (68) follows. □

Note that the external force $f(x,t) = ct^{\gamma-1}E_{\alpha,\gamma}(-bt^{\alpha})$ goes to zero for $t \to 0$ since $\gamma > 1$. On the other side by using the asymptotic behavior of the Mittag–Leffler function $E_{\alpha,\beta}(\tau) \sim -\frac{\tau^{-1}}{\Gamma(\beta-\alpha)}$ for $\tau \to \infty$ [18] the external force shows the behavior $\frac{c}{b\Gamma(\gamma-\alpha)}t^{-\alpha+\gamma-1}$ which goes to zero since $\alpha - \gamma + 1 > 0$. From solution (68) it can be concluded that in the long time limit ($t \to \infty$) a power-law decay can also be shown.

Example 1. It can be easily shown, by using solution (63), that following time fractional partial differential equation

$$D_*^{\alpha}u(x,t) = a^2\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{74}$$

$$u(x,t)|_{x=0} = 0, \qquad u(x,t)|_{x=2} = 0, \tag{75}$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+} = 0, \qquad u(x,0+) = 0.03\cdot x(2-x), \tag{76}$$

where $1 < \alpha \leqslant 2$ and $0 \leqslant x \leqslant 2$, has a solution of the form

$$u(x,t) = \frac{0.96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}E_{\alpha}\left[-\frac{(2n-1)^2\pi^2a^2}{4}t^{\alpha}\right]\sin\left[\frac{(2n-1)\pi x}{2}\right]. \tag{77}$$

Note that in the case when $\alpha = 2$, solution (77) has the following form [32]:

$$u(x,t) = \frac{0.96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\left[\frac{(2n-1)\pi a}{2}t\right]\sin\left[\frac{(2n-1)\pi x}{2}\right]. \tag{78}$$

Solution (78) for $a = 1$ is given in figure 1.

Remark 1. Solution (18) is also applicable to the differential equation of form (60) for $0 < \alpha \leqslant 1$, boundary conditions $u(x,t)|_{x=0} = 0$ and $u(x,t)|_{x=l} = 0$, and an initial condition

$u(x, 0+) = g(x)$. This equation represents a time fractional diffusion (or heat conduction) differential equation, and in cases where $\alpha = 1$ one obtains the well-known result

$$u(x,t) = \sum_{n=1}^{\infty} c_n \,\mathrm{e}^{-\frac{n^2\pi^2a^2}{l^2}t} \sin\left(\frac{n\pi x}{l}\right), \tag{79}$$

where c_n is given by (57).

4. Conclusion

We investigate an exact solution of a general time fractional wave equation for a vibrating string. The solution of the equation is expressed in terms of the Mittag–Leffler-type functions, integral operator (14) and complete set of eigenfunctions of the Sturm–Liouville problem. It is shown that equation (1) is a generalization of the general wave equation for a vibrating string, whose solution follows from theorem 1 for $\alpha = 2$. Some special cases of equation (1) are considered and it is shown that the corresponding solution of integer order equations are special cases of the time fractional equations.

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